exploratory investigations and data analyses, such as where the need for curve fitting often occurs. The statistical analysis in these cases is generally referred to as significance testing, or significance evaluation, and, in contrast to the author's use of the term, this refers to the process of calculating a significance level from the data and reporting this as a summary statistic which measures the consonance of the data with some hypothesized model. Subsequent decisions may be based at least in part on observed significance levels, but the point is that one is not required beforehand to specify all hypotheses and risk levels. To do so is contrary to the nature of scientific investigation and, as the author pointed out, often involves an unsatisfactory element of arbitrariness. Much statistical literature leaves the unfortunate impression that the number one problem facing statisticians is the choice of the Type I error level, but that it has been solved for them since time immemorial by taking this risk to be 0.05 or 0.01. Those texts which mention significance tests and contrast them with tests of hypotheses, often do so with a note of apology for their lack of mathematical rigor. This is unfortunate. Users of statistics need to be aware that statistics can be used for learning from data as well as for making decisions which have a given probability of success under certain idealized conditions. A more extensive discussion of this matter is given elsewhere.[2-4]

III. The Distribution of B

The quantity B is a statistic, a function of data. For it to be of use, it is necessary to have some scale of measurement to judge it against. Only then can we tell how large a large B value is, or how much larger $B = 9.19$ is than $B = 1.58$. We now consider a statistical scale of measurement, which is the probability distribution of B under sampling from a given model.

B is always at least 1.0, so we can write it as

$$B = S'/S'' = 1 + R'/S''$$

where $R' \geqq 0$. Consider the polynomial models

$$Y_1 = \beta_{10} + \beta_{11}X + \beta_{12}X^2 + \ldots + \beta_{1K}X^K + e$$
$$Y_2 = \beta_{20} + \beta_{21}X + \beta_{22}X^2 + \ldots + \beta_{2K}X^K + e$$

where the subscripts, 1 and 2, refer to the two halves of the data, and the hypothesis

$$\beta_{10} = \beta_{20}; \quad \beta_{11} = \beta_{21}; \ldots; \quad \beta_{1K} = \beta_{2K}$$

that is, consider the case, which one might label minimum bias, where the two-data halves are generated by the same model. In this case, it can be shown (see Graybill[5]) that

$$F = \frac{R'/(K+1)}{S''/(N-2K-2)} = \frac{(B-1)(N-2K-2)}{(K+1)}$$

has an F distribution with $K+1$ and $N-2K-2$ degrees of freedom. Departures from this hypothesis tend to inflate R', and hence B, so one can quantify the comparison of observed data to this model by calculating

$$P = \text{Prob}[F(K+1, N-2K-2) \geqq \text{observed } F]$$

that is by making a test of significance of the closeness of the data to the minimum bias hypothesis. [$F(\cdot,\cdot)$ is an F random variable with the indicated degrees of freedom.] The larger P is, the closer, or more in agreement, the data are to the hypothesis. Note that this criterion takes into consideration the sample size and the degree of the polynomial, neither of which enter into the author's comparison of B-values. The simplest case, $K = 0$, is just the usual test for equality of two means and realizing this helps clarify just what is being measured by B.

IV. Example

Consider the illustrative example of the author and his Figs. 4–8. Table 1 gives B, F, the degrees of freedom, and P for polynomials of degree 0–4. Note that the minimum B in this case agrees with the maximum P. However, both the second and fourth degree polynomials, with $B = 1.95$ and 9.19, respectively, yield larger P-values than the linear fit which has $B = 1.58$. That is, the B's for the second and fourth degree curves are not as unusual under the minimum bias model as the smaller B for the linear fit. Thus, comparing B-values as though they were points on the real line could be quite misleading.

Table 1 Analysis of sample data

Degree	B	F	Degrees of freedom	P
0	6.31	53.1	(1, 10)	$\approx 10^{-5}$
1	1.58	2.33	(2, 8)	0.16
2	1.95	1.90	(3, 6)	0.23
3	1.43	0.43	(4, 4)	0.78
4	9.19	3.28	(5, 2)	0.25

References

[1] Dylewski, T. J., "Criteria for Selecting Curves for Fitting to Data," *AIAA Journal*, Vol. 8, No. 8, Aug. 1970, pp. 1411–1415.

[2] Easterling, R. G., "A Personal View of the Bayesian Controversy in Reliability and Statistics," *IEEE Transactions on Reliability*, Vol. R-21, No. 3, Aug. 1972, pp. 186–194.

[3] Fisher, R. A., *Statistical Methods and Scientific Inference*, 1st ed., Hafner, New York, 1956, pp. 37–78.

[4] Kempthorne, O. and Folks, J. L., *Probability, Statistics and Data Analysis*, Iowa State University Press, Ames, Iowa, 1971, pp. 312–362.

[5] Graybill, F. A., *An Introduction to Linear Statistical Models*, Vol. 1, McGraw-Hill, New York, 1961, pp. 133–140.

Reply by Author to R. G. Easterling

T. J. Dylewski
Lockheed Missiles and Space Company, Inc., Sunnyvale, Calif.

THE part of the Comment headed "Statistical Tests" expresses generalized personal attitudes towards uses of statistical tests and their nomenclature. This is a popular debating topic for some statisticians who, nevertheless, often seem to carry out their professional activities not much differently or more ably than their nominal opponents. This material has only incidental pertinence to my paper.

It can be inferred from the rest of the comment that the paper was not carefully read. I did not, for example, state that the bias-ratio, B, does not permit a test of significance; I did say that such a test is not available. I did not claim the lack of a test of significance as an advantage; I did say that the bias-ratio criterion has the advantage of not requiring the selection of a significance level.

More importantly, the comment presents an erroneous derivation of the distribution of B in an attempt to calculate its significance. This is done by an unjustifiable transformation of the B-distribution into the F-distribution.

I had investigated the possibility of using the variance-ratio test based on the F-distribution prior to devising the bias-ratio criterion. Some of the reasons for discarding the former are stated in the paper. Since the commenter appears to have overlooked them, I will restate them.

The F-distribution is applicable to normally-distributed random deviations. This condition is usually realized adequately closely only when a function without fitting bias is found. For any other function, howsoever chosen for "best-fit," the deviations may be far from normal and random. Moreover, the F-distribution applies to one test only, not to a sequence of tests made for the purpose of locating an extreme probability. Such

Received July 26, 1973.

Index categories: Computer Technology and Computer Simulation Techniques; Research Facilities and Instrumentation.

* Senior Staff Scientist.

use is grossly incorrect, since it disregards an essential additional parameter—the number of different curves fitted.

Use of the variance-ratio test instead of the bias-ratio criterion tends to result in the selection of functions with too many constants. Being right for the wrong reasons, the comment indicates the cubic as most suitable for the example in the paper. The second-best choices are more revealing about the two approaches. The paper selects the first-degree polynomial while the comment prefers the fourth. Even a visual examination should suffice for determining which of them has subset regression curves that might more reasonably be extrapolated to the regions to which they had not been fitted.

For curves that oversmooth data, B is a maximum-likelihood estimator of the ratio of the sum of squares of deviations for a given curve to the sum of squares of deviations for the most suitable, unbiased, even though unknown curve. For the oversmoothed condition, B is measured on a true ratio scale, which admits all arithmetical and statistical operations. In particular, the distance (B-1) is then a measure of the residual, potentially removable bias in terms of irremovable random variation.

The bias-ratio criterion has been in frequent use with various kinds of data, and no difficulty has arisen from the fact that probabilities are not available for the B-s. While some readers might feel more at ease in the probability domain, I know of no valid way of getting there at present.

Comment on "Analysis of Embedded Shock Waves Calculated by Relaxation Methods"

Tsze C. Tai*
Naval Ship Research and Development Center, Bethesda, Md.

Murman has introduced a new finite-difference operator for calculating embedded shock waves by relaxation methods.[1] It constitutes a significant refinement of the earlier version of the relaxation scheme developed by Murman and Cole.[2] The purpose of this Comment is to substantiate a rationale for the difference in shock strength and shock location generated by using the new and old "shock-point" operators.

Although there might still be some question regarding the consistency condition on the differential equation, the use of the new shock-point operator, Eq. (11) of Ref. 1, is acceptable to the extent that it satisfies the integral equation. The Murman numerical results show that the solution using the new operator —fully conservative relaxation (FCR)—has a stronger shock and one that is considerably farther aft than obtained by using the old elliptic operator—not fully conservative relaxation (NCR). The trend has been consistent throughout all the data presented. Murman has given no reason for such differences or no discussion other than some equivocal remarks on the attribution to the re-expansion singularity. Of course, complaints about the experimental tests or small disturbance equations are irrelevant to the present issue.

In fact, the situation whereby different shock strength and location computed by pure inviscid theory but with different shock relations are encountered is mathematically rigorous and physically realistic. This was experienced at NSRDC when the effect of entropy change across an embedded shock wave was

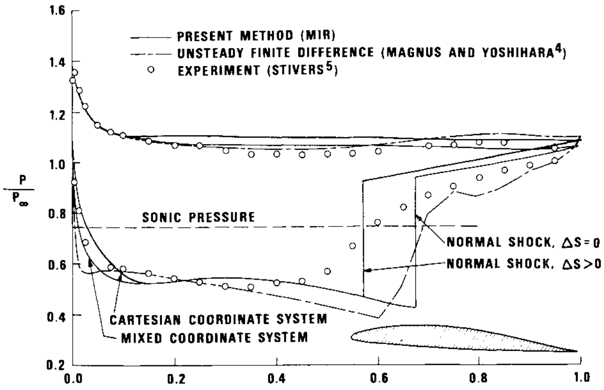

Fig. 1 Pressure distribution on an NACA 64A410 airfoil at $M_\infty = 0.72$ and $\alpha = 4°$.

evaluated.[3] However, the numerical scheme used there was the method of integral relations rather than the finite-difference technique. In the case of a finite increase in entropy across the the shock wave, it was found that the shock strength was weakened and the shock location moved forward as shown in Figs. 1 and 2. These figures are taken from Tai[3] (originally Figs. 6 and 7) and include data from Magnus and Yoshihara,[4] Stivers,[5] Steger and Lomax,[6] and Graham et al.[7] The increase in entropy corresponds to the decrease in total pressure. A simple relation between the total pressure and entropy change can be found in Liepmann and Roshko.[8] Mathematically, the inclusion of entropy change provides a rigorous inviscid model to account for the rotational term in the full inviscid flow equations. Physically, the change of entropy creates vorticity behind the shock wave; this has a cumulative effect in the far downstream and consequently feeds back to the shock itself.[3]

As a matter of fact, the spurious sinklike terms of Eq. (18b) of Ref. 1 have an effect similar to a decrease in total pressure or an increase in entropy. Since the old elliptic operator is solely responsible for the rise of these sinklike terms, it is apparent that its use gives a weaker and earlier shock wave than one computed by the new shock-point operator. It is interesting to note that the unsteady finite-difference results, which agree well with the solution obtained by using the FCR method in Ref. 1, correspond to the potential flow solution ($\Delta S = 0$) in Fig. 1. On the other hand, those of the relaxation method based on the previous version lie close to the nonisentropic solution (normal shock, $\Delta S > 0$) in Fig. 2. This provides further evidence that the new FCR method has the capability of representing the true potential flow solution.

In any case, the similar effect obtained by using the old elliptic operator and incorporating the entropy change is just a coincidence; their mathematical bases are completely different.

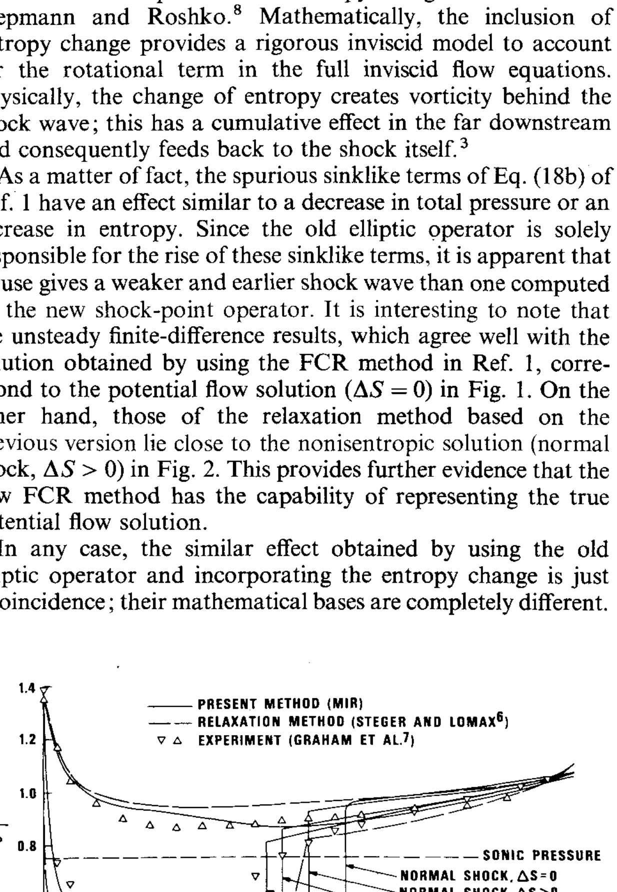

Fig. 2 Pressure distribution on an NACA 0015 airfoil at $M_\infty = 0.729$ and $\alpha = 4°$.

Received October 19, 1973; revision received November 30, 1973.
Index category: Subsonic and Transonic Flow.

* Aerospace Engineer. Member AIAA.